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TRANSITION CURVES.

By W. B. Lee, M. Am. Soc. C. E.

WITH DISCUSSION.

For ordinary railroad purposes, the transition curves in common use are of about equal utility. All require tables, and, with these in hand, can be readily staked out. When, however, we are confronted with the task of designing and detailing for the shop, track-work such as turnouts and crossings, new problems arise. The multiple compound curve requires a maze of complicated computation when applied to this class of work (in which loci are absolutely necessary). The cubic parabola appeals to the writer as the simplest curve which will fulfill the conditions of a transition. But, it has been urged by some that the cubic parabola is not a true transition curve, nor can it be readily staked out with a transit.

Let us see, then, if we can more nearly approximate these prerequisites in a curve of similar character.

In developing the cubic parabola the deviation from exact values consists in substituting the differential of the abscissa for the differential of the length of the curve, in the general formula for the radius of curvature,

$$r = \frac{d s^3}{d x \, d^2 y} = \frac{d x^3}{d x \, d^2 y} = \frac{d x^2}{d^2 y}.$$

Now, if we reverse the substitution we can say

$$\frac{d\,s^3}{d\,s\,d^2\,y} = \frac{d\,s^2}{d^2\,y}.$$

Equating with the other value of $r = \frac{R\,s_1}{s}$, and inverting, we have

$$\frac{d^2\,y}{d\,s^2} = \frac{s}{R\,s_1},$$

whence, by integrating twice, we have $y = \frac{s^3}{6\,R\,s_1}$.

This is more nearly the equation of a true transition curve, for $r = \frac{R\,s_1}{s}$ (which fulfills the condition of a transition curve, and in which R is the radius of the circular curve to be eased, and s_1 the total length of the transition curve) is now exact, and the substitution of $d\,s$ for $d\,x$ involves less error than the substitution of $d\,x^3$ for $d\,s^3$. Moreover, since it is expressed in terms of the length of the curve, it is better adapted to staking out with a transit. Before the deflection angles can be readily obtained, x must be determined. This cannot be found exactly in a finite number of terms, but, by a slight approximation, a very convenient and practical formula can be developed.

In triangles of small angle between hypothenuse and base we may express the excess of the former over the latter as follows:

Let e = excess; p = perpendicular; h = hypothenuse; and b = base.

Then $e = \frac{p^2}{2\,h}$ (nearly), for $h^2 - b^2 = p^2$, but $b = h - e$, whence $h^2 - (h^2 - 2\,h\,e + e^2) = p^2$ and $e = \frac{p^2}{2\,h - e}$; dropping $-\,e$, as relatively very small, we have $e = \frac{p^2}{2\,h}$ (nearly) ....(Q. E. D.)

Call the total excess of s over x, t, that is, $x = s - t$.

Then $$d\,t = \frac{d\,y^2}{2\,d\,s}.$$

Now $$y = \frac{s^3}{6\,R\,s_1}, \quad d\,y = \frac{s^2\,d\,s}{2\,R\,s_1}, \quad d\,y^2 = \frac{s^4\,d\,s^2}{4\,R^2\,s^2_1}.$$

Substituting, $$d\,t = \frac{s^4\,d\,s}{8\,R^2\,s_1^2}.$$

Integrating, $$t = \frac{s^5}{40\,R^2\,s^2_1}, \text{ and } x = s - \frac{s^5}{40\,R^2\,s^2_1}.$$

$\frac{s^5}{40\,R^2\,s_1^2}$ may be obtained on the slide rule to the one-hundredth

part of a foot. Any value of x being now at hand we may determine any deflection angle B by

$$\frac{y}{x} = \tan. B.$$

The total angle A which is made by a tangent to the curve at any point with the initial tangent is found as follows:

$$\frac{dy}{ds} = \sin. A,$$

$$y = \frac{s^3}{6 R s_1},$$

$$\frac{dy}{ds} = \frac{s^2}{2 R s_1} = \sin. A.$$

This curve has the further advantage that, after we have fixed the constant $R s_1$, we can find any r (by dividing the constant by s) with a greater degree of accuracy than in the ordinary cubic parabola.

For those who would still prefer the equation $y = \frac{x^3}{6 R x_1}$, the curve may be rectified by the following formula:

$$s = x + \frac{x^5}{40 R^2 x_1^2}.$$

It is found in the same manner as

$$x = s - \frac{s^5}{40 R^2 s_1^2}.$$

Notwithstanding the substitute which he has offered, the writer would still urge upon those engaged in designing layouts of track-work the use of the equation $y = \frac{x^3}{6 R x_1}$, on account of the facility with which intersections may be determined by it. For example, suppose we have two transition curves leaving a common tangent and intersecting, as in Fig. 1.

$$y_1 = y_2, \quad x_1 + x_2 = 100.$$

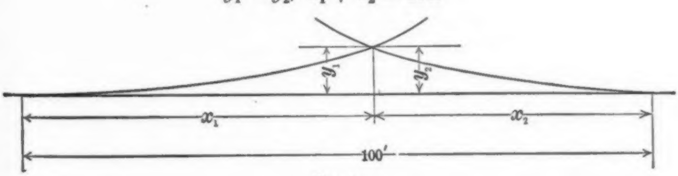

FIG. 1.

To simplify formulas let us call $R\,x_1$, the constant quantity in the equation of the curve, C_1 and C_2,

then $$y_1 = \frac{x_1^{\,3}}{6\,C_1} = y_2 = \frac{x_2^{\,3}}{6\,C_2};$$

and $$x_1 = x_2 \sqrt[3]{\frac{C_1}{C_2}};$$

substituting in $x_1 + x_2 = 100$,

$$x_2 \sqrt[3]{\frac{C_1}{C_2}} + x_2 = 100,$$

and $$x_2 = \frac{100}{\sqrt[3]{\frac{C_1}{C_2}} + 1}$$

after finding x_1, y_1 and y_2, we may find the total angle, A, for each curve, by tan. $A = \frac{x^2}{2\,C}$, and their sum is the angle of intersection.

DISCUSSION.

GEORGE D. SNYDER, M. Am. Soc. C. E. (by letter).—This paper is a valuable addition to the mathematics of transition curves, and the formulas given are of great practical utility, particularly in street railway work. It is unfortunate that, notwithstanding the uniform practice which exists in this country in regard to staking out railway curves, such a wide diversity should exist in the use or non-use of transition curves. In spite of the various articles which have appeared in the technical press, and the discussions by engineering societies, no general use of transition curves, or general uniformity of practice, has resulted. Mr. Snyder.

All agree as to the benefits to be derived from the use of such curves, and many valuable methods of staking them out have been published, but, for some reason or other, the practical men who are called on to introduce them seem to fight shy of them as a needless refinement.

A comparison of the various methods will show that most of them differ but little, and give results that are practically uniform, within reasonable limits. The formulas are generally simple, and the application not complicated. In most methods the necessary information for use in the field has been tabulated for convenience, and the fact that these tables are more or less complicated, and limited in scope, may have deterred some from using them. As the necessary calculations are not complicated, it would seem that the use of tables is unnecessary. The author refers to the use of the slide rule for such calculations, and the writer is of the opinion that, if the use of this instrument was more general, a more general use of transition would result, as with its aid all the calculations necessary for a transition curve can be made in a few minutes, thus making tables unnecessary.

The author's formulas are well adapted for use on the slide rule. They can be applied to a method of deflections or offsets, the offset method being generally more convenient for railroad work, for the reason that the field work, field notes and plotting of the curve are but little changed from the methods in use for ordinary curves; also, for the reason that the transition curve need not be staked out in detail during the preliminary stages of the work, but merely provided for by offsetting the circular curve, toward its center from the tangents, leaving the staking of the transition curve until ready for the laying of the track.

On railroad work, using the offset method, the information required for any given degree of curve is, the length of transition curve, the offset from the tangent to the main curve when prolonged to a parallel

Mr. Snyder. tangent at the *P. C.*, and the intermediate offsets to various points on the transition curve. For this purpose the author's formula becomes:

$$Y = \frac{S^2}{24\,R}$$

In which $Y =$ the offset to the main curve at the *P. C.*

By putting

$$\frac{5\,730}{D^\circ} = R,$$

in which D° is degree of curve, we get

$$*\,Y = \frac{S^2\,D^\circ}{137\,520}$$

and

$$S = \sqrt{\frac{137\,520\,Y}{D^\circ}}$$

With this formula, by a single setting on the slide rule, the offset for any length of transition curve for any given degree of curve can be obtained to the nearest one-hundredth of a foot, at a glance. Intermediate offsets, being as the cube of the distance, can likewise be obtained readily on the slide rule.

The transition curve bisects the offset Y at the *P. C.*, and is bisected by the offset line at this point. It will be found most convenient in practice to divide the half length of the transition curve into a number of equal parts, and calculate on the slide rule the offsets for a corresponding number of points. Points can then be set on half the transition curve by offsets from the tangent, and on the remaining half by offsets from the circular curve, using the same offsets in inverse order and opposite direction.

If the deflection method is preferred, the deflections, being as the square of the distance, can likewise be obtained readily on the slide rule. The preceding methods and formulas can be applied with equal facility to compound and reverse curves, and to the improvement of old lines, as well as on new construction.

The writer used similar methods on the introduction of transition curves on more than 100 miles of existing railway, and, while the methods used were not mathematically exact, no errors sufficient to be appreciable manifested themselves in the field work. In this work, stone monuments were set to mark the beginning, end and middle points of each transition curve, intermediate points being set about 25 ft. apart and marked by oak stakes.

The cost of this work may be of interest. The most expensive work, naturally, was where there was the most curvature, which was on a section about 20 miles long, 47% being curved, the average de-

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**Transactions*, Am. Soc. C. E., Vol. xxxvi, p. 511.

gree of curve being 5° 00′ and the amount of curvature being 125° 05′ per mile. The cost per mile was as follows: Mr. Snyder.

Adjusting and marking center line	\$17.17
Stone monuments	3.60
Labor, planting and distributing monuments	7.50
Total	\$28.27

For this sum per mile a railroad can put the easing of its curves on a more scientific basis than the "crow-bar" and "rule-of-thumb" methods of the track foreman, and at the same time obtain much better results.

Some roads are expending large sums for the elimination of curvature, and many others only refrain from doing so for lack of funds. The latter may find the introduction of transition curves a profitable investment, pending the time when they can afford to eliminate their excessive curves entirely.

C. A. SUNDSTROM, M. Am. Soc. C. E. (by letter).—Transition curves are of great importance to the railroad engineer, and are gradually coming into more general use. These curves are also of great importance to the traveling public. For, if a railroad is provided with transition curves, there can be no shock and subsequent inconvenience to the passengers when the train is entering or leaving a curve. But, although the cubic parabola is the only curve that fully meets the conditions of a transition curve, the spiral is more commonly used. There can be no other reason for this, than that the majority of railroad engineers imagine that it is impossible to stake out a cubic parabola by means of a transit. The writer has had a great deal of experience with transition curves, and nearly twenty years ago worked up tables which will enable the engineer to stake out a transition curve, derived from the cubic parabola, just as easily as a circular curve. Mr. Sundstrom.

Before showing the use of the tables, it will be necessary to review briefly some of the theories upon which they are based.

The equation of the cubic parabola is

$$y = \frac{x^3}{6Rl},$$

in which R is the radius of the center curve, and l the length of the parabolic arc.

This being the equation of the curve obtained when a cantilever beam bends under the action of a load suspended at the free end, the cubic parabola is also called the elastic curve.

The length of the parabolic arc depends upon the elevation of the outer rail and the grade assumed in order to reach or leave the

Mr. Sundstrom. elevation. The elevation, h, of the outer rail depends upon the gauge G, the velocity v, and the radius of curvature R.

Thus $$h = \frac{G\,v^2}{g\,R} = \frac{G\,v^2}{36.16\,R}.$$

Assuming a grade of 1 ft. in n feet, the length of the parabolic arc will be

$$l = n\,h = \frac{n\,G\,v^2}{32.16\,R} = \frac{C}{R},$$

$$C \text{ denoting } \frac{n\,G\,v^2}{32.16}.$$

By differentiating the equation of the cubic parabola we obtain

$$\frac{d\,y}{d\,x} = \frac{x^2}{2\,Rl},$$

which is the natural tangent of the angle which the tangent to any point of the curve makes with the tangent of the curve.

The deflections are obtained by considering the triangle $A\ B\ C$, in Fig. 2, in which $A\ D\ B$ is the parabolic arc, the tangent of which is $A\ C$. The deflection angle, $D\ A\ C$, or d, is obtained from

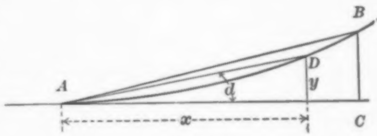


FIG. 2.

$$\tan.\ d = \frac{y}{x} = \frac{x^2}{6\,R\,l} = \frac{x^2}{6\,C}.$$

Thus the deflection is a function of the distance only, and entirely independent of the main curve.

The deflection for the point of osculation is obtained from

$$\tan.\ d = \frac{l^2}{6\,R\,l} = \frac{l^2}{6\,C}.$$

The tables have been calculated in conformity with the following conditions: That the grade of the outer rail rises 1 ft. in 300 ft.; the velocity is 40 miles an hour, and the gauge is 4 ft. 9 ins. Consequently, the constant $C = 152314.74$. These tables can be used also for a speed of 60 miles per hour, but, in that case, the grade of the outer rail rises 1 ft. in 133.3 ft.

The tables are best illustrated by an example.

Let $A\ B$ and $A\ C$ (Fig. 3) be two tangents, intersecting at A; the angle of intersection $P\,A\,B = 120°$. Given, a 6° curve, to determine the elastic approaches.

The first step is to ascertain whether the intersection angle is large enough for a 6° curve. Mr. Sundstrom.

According to Table No. 1 the tangential angle for a 6° curve is 4° 46′. This is the angle $K\ F\ E$. As the angle $S\ I\ F = P\ A\ B - 2\ (K\ F\ E)$, it is evident that, when the angle $P\ A\ B = 2\ (K\ F\ E)$, the angle $S\ I\ F$ is equal to 0. In that case the points N and E coincide, and the curve consists of two elastic approaches only. When $2\ (K F E)$ is larger than the intersection angle $P\ A\ B$, the center curve must be changed to a degree of curvature having a sufficiently small tangential angle.

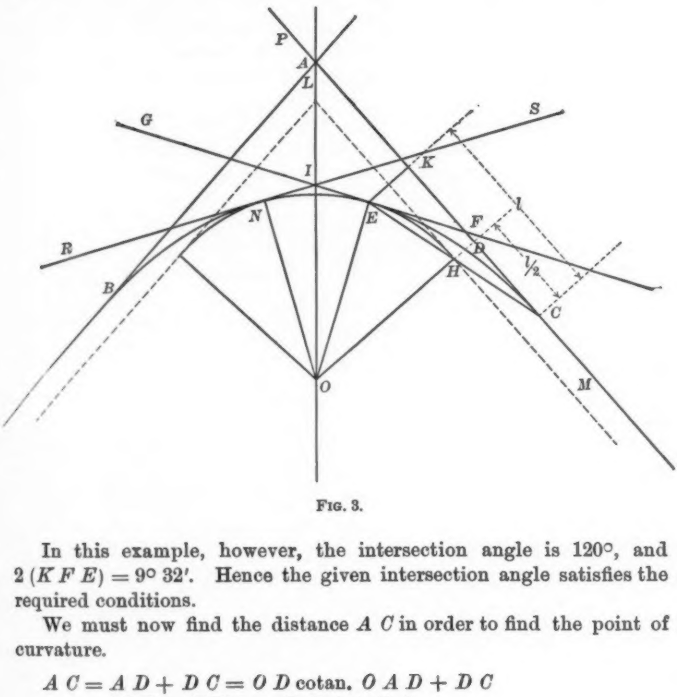


FIG. 3.

In this example, however, the intersection angle is 120°, and $2\ (K\ F\ E) = 9°\ 32'$. Hence the given intersection angle satisfies the required conditions.

We must now find the distance $A\ C$ in order to find the point of curvature.

$$\begin{aligned} A\ C &= A\ D + D\ C = O\ D \text{ cotan. } O\ A\ D + D\ C \\ &= (O\ H + H\ D) \text{ tan. } D\ O\ A + D\ C \\ &= (R + s) \text{ tan.} \frac{I}{2} + \frac{l}{2}; \end{aligned}$$

R being the radius of the center curve; s the corresponding shift, $D\ H$; I the intersection angle, and l the length of the parabolic arc, or elastic approach.

Mr. Sundstrom. We have, therefore,

$$A\ C = (955.4 + 1.11) \text{ tan. } \frac{120^\circ}{2} + \frac{159.4}{2} = 1736.4.$$

The values of l and s are taken from Table No. 1 for a 6° curve.

After the distances $A\ C$ and $A\ B$ have been measured, set the transit at C and sight on A, the vernier being set at zero. If stakes are to be set every 25 ft., the first deflection must be for that distance. According to Table No. 2, the deflection for 25 ft. is 3′, for 50 ft. 10′, for 75 ft. 22′, for 100 ft. 37′, for 125 ft. 59′, for 150 ft. 1° 25′. The length of the elastic approach for a 6° curve, according to Table No. 1, is 159.4 ft., which makes the last deflection 1° 36′. We have now reached the point of osculation. Set the transit over the point of osculation, E, and lay off the tangent, $F\ E\ G$; the tangential angle, $G\ F\ A$, being 4° 46′. The last operation is best accomplished in the following manner:

The angle $F\ E\ C = G\ F\ A - E\ C\ K = 4^\circ\ 46' - 1^\circ\ 36' = 3^\circ\ 10'$. The angle $C\ E\ G$, being the supplement of the angle $F\ E\ C$, is equal to 176° 50′. After the transit has been placed over the point E, set the vernier at 176° 50′, take a sight at C, and turn the instrument to zero. We have now obtained the new tangent, and are ready to lay out the main curve. Before this can be done, however, the intersection angle, $G\ I\ R$, must be found. From what precedes, this angle is equal to $P\ A\ B - 2\ (K\ F\ E) = 120^\circ - 9^\circ\ 32' = 110^\circ\ 28'$, and we stake out a 6° curve for this angle in the usual manner. After the point N is reached, move the instrument to B, and proceed in the same manner with the elastic approach $B\ N$.

When the intersection angle is small, that is to say, a trifle more than twice the tangential angle, the main curve can be shifted and provided with elastic approaches by simply furnishing the point of curve, the point of tangent, the points of osculation and the external secant. The co-ordinates for the point of osculation are obtained from Table No. 1.

Questions have arisen concerning the accuracy of the assumption that the length of the parabolic arc is equal to the abscissa of its terminus. It must be admitted, of course, that there is a slight difference. In order to ascertain the exact amount of this difference, a formula for the rectification of a cubic parabolic arc will be derived:

The length of any arc is $L = \int \sqrt{1 + \frac{d y^2}{d x^2}}\ d x$

The equation of the cubic parabola is

$$y = \frac{x^3}{6\ C},$$

hence

$$\frac{d y}{d x} = \frac{x^2}{2\ C},$$

Mr. Sundstrom.

and
$$\frac{d\,y^2}{d\,x^2} = \frac{x^4}{4\,C^2}.$$

Thus
$$L = \int \left(1 + \frac{x^4}{4\,C^2}\right)^{\frac{1}{2}} d\,x$$
$$= \int \left(1 + \frac{x^4}{8\,C^2} - \frac{x^8}{32\,C^2} + \text{etc.}\right) d\,x.$$

As C is very large in comparison with x, the third term in the series can be discarded, and
$$L = x + \frac{x^5}{40\,C^2} + C_1,$$
for
$$x = 0, \quad C_1 = 0;$$
hence
$$L = x + \frac{x^5}{40\,C^2}.$$

Examining the length of a parabolic arc for an abscissa of 100 ft. it is found that
$$L = 100 + \frac{(100)^5}{40 \times (152\,314)^2},$$
or, $L = 100 + 0.011 = 100$ ft. $0\frac{1}{8}$ in.

For an abscissa of 150 ft. the arc is equal to 150 ft. 1 in.

For an abscissa of 186 ft., the largest one in the tables, the arc will be 186 ft. $2\frac{7}{8}$ ins.

For an abscissa of 200 ft., the arc is equal to 200 ft. $4\frac{1}{8}$ ins.

These figures show that, for practical purposes, the arc may be assumed to be equal to the abscissa of its terminus. Where shop work is required, the longer arcs ought to be rectified.

The foregoing refers only to the tables which have been calculated for steam railroad purposes. In the curves for street railways, as will be seen from the tables, the arcs have been rectified.

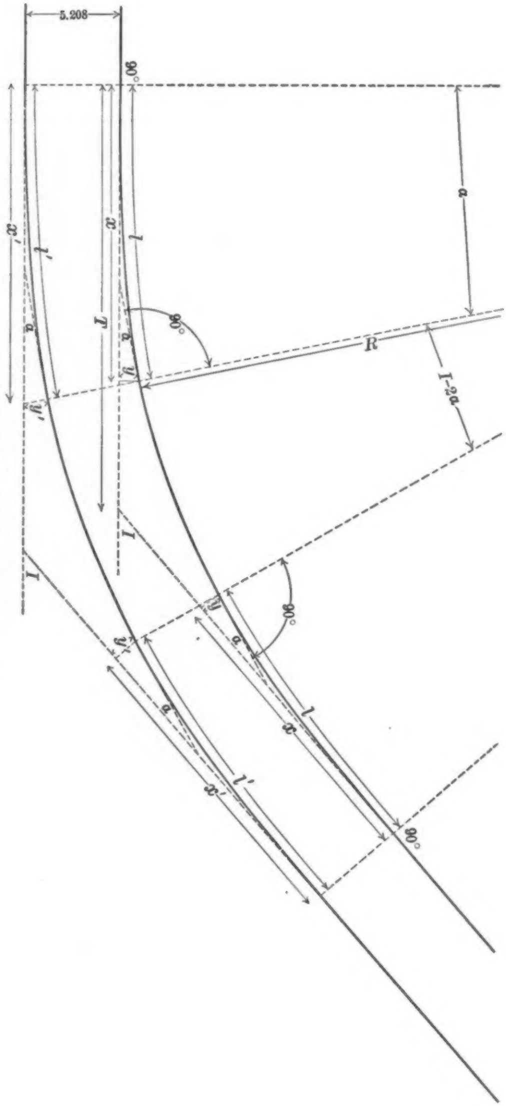
The writer has frequently had occasion to run in transition curves on street railroads, and then used the figures in Table No. 3. The convenience of this table is self-evident, and its use presents no difficulties in the shop, as the same templet can be used for all approaches, the only variable quantity being the length.

Mr. Sundstrom.

TABLE No. 1.

Degree of curvature.	Length of parabolic arc.	Shift.	Ordinate in point of osculation.	Tangential angle in point of osculation.
3°	79.7	0.14	0.55	1° 12′
5′	82.0	0.15	0.60	1° 16′
10′	84.2	0.16	0.65	1° 20′
15′	86.4	0.18	0.71	1° 24′
20′	88.6	0.19	0.76	1° 29′
25′	90.8	0.20	0.82	1° 33′
30′	93.0	0.22	0.88	1° 38′
35′	95.2	0.24	0.95	1° 42′
40′	97.5	0.25	1.02	1° 47′
45′	99.7	0.27	1.08	1° 52′
50′	101.9	0.29	1.16	1° 57′
55′	104.1	0.31	1.24	2° 2′
4°	106.3	0.33	1.32	2° 7′
5′	108.5	0.35	1.40	2° 13′
10′	110.7	0.37	1.49	2° 18′
15′	113.0	0.39	1.58	2° 24′
20′	115.2	0.42	1.67	2° 30′
25′	117.4	0.44	1.77	2° 35′
30′	119.6	0.47	1.87	2° 41′
35′	121.8	0.49	1.98	2° 47′
40′	124.0	0.52	2.09	2° 53′
45′	126.2	0.55	2.20	3°
50′	128.5	0.58	2.32	3° 6′
55′	130.7	0.61	2.44	3° 13
5°	132.9	0.64	2.57	3° 19′
5′	135.1	0.67	2.70	3° 26′
10′	137.3	0.71	2.83	3° 32′
15′	139.5	0.74	2.97	3° 39′
20′	141.7	0.78	3.12	3° 46′
25′	143.9	0.82	3.26	3° 53′
30′	146.2	0.85	3.42	4° 1′
35′	148.4	0.89	3.57	4° 8′
40′	150.6	0.93	3.74	4° 15′
45′	152.8	0.98	3.90	4° 23′
50′	155.0	1.02	4.08	4° 31′
55′	157.2	1.06	4.25	4° 38′
6°	159.4	1.11	4.44	4° 46′
5′	161.6	1.16	4.62	4° 54′
10′	163.9	1.20	4.81	5° 2′
15′	166.1	1.25	5.01	5° 10′
20′	168.9	1.30	5.22	5° 19′
25′	170.5	1.36	5.42	5° 27′
30′	172.7	1.41	5.64	5° 36′
35′	174.9	1.46	5.86	5° 44′
40′	177.1	1.52	6.08	5° 53′
45′	179.3	1.58	6.31	6° 2′
50′	181.5	1.64	6.55	6° 11′
55′	183.8	1.70	6.79	6° 20′
7°	186.0	1.76	7.04	6° 29′

Mr.Sundstrom.



TRANSITION CURVES FOR STREET RAILWAYS

Fig. 4.

Mr. Sundstrom.

TABLE No. 2.

Deflection.	Distance.	Deflection.	Distance.	Deflection.	Distance.
1'	16.3	52'	117.6	43'	165.5
2'	23.1	53'	118.7	44'	166.3
3'	28.2	54'	119.8	45'	167.1
4'	32.6	55'	120.9	46'	167.9
5'	36.5	56'	122.0	47'	168.7
6'	39.9	57'	123.1	48'	169.5
7'	43.1	58'	124.2	49'	170.3
8'	46.1	59'	125.2	50'	171.0
9'	48.9	1°	126.3	51'	171.8
10'	51.6	1'	127.4	52'	172.6
11'	54.1	2'	128.4	53'	173.4
12'	56.5	3'	129.4	54'	174.1
13'	58.8	4	130.4	55'	174.9
14'	61.0	5'	131.5	56'	175.6
15'	63.1	6'	132.5	57'	176.4
16'	65.2	7'	133.5	58'	177.2
17'	67.2	8'	134.5	59'	177.9
18'	69.2	9'	135.4	2°	178.6
19'	71.1	10'	136.4	1'	179.4
20'	72.9	11'	137.4	2'	180.1
21'	74.7	12'	138.3	3'	180.9
22'	76.5	13'	139.3	4'	181.6
23'	78.2	14'	140.3	5'	182.3
24'	79.9	15'	141.2	6'	183.1
25'	81.5	16'	142.2	7'	183.8
26'	83.1	17'	143.1	8'	184.5
27'	84.7	18'	144.0	9'	185.2
28'	86.3	19'	144.9	10'	185.9
29'	87.8	20'	145.8	11'	186.6
30'	89.3	21'	146.7	12'	187.4
31'	90.8	22'	147.7	13'	188.1
32'	92.2	23'	148.6	14'	188.8
33'	93.7	24,	149.5	15'	189.5
34'	95.1	25'	150.3	16'	190.2
35'	96.5	26'	151.2	17'	190.9
36'	97.8	27'	152.1	18'	191.6
37'	99.2	28'	153.0	19'	192.3
38'	100.5	29'	153.8	20'	193.0
39'	101.8	30'	154.7	21'	193.6
40'	103.1	31'	155.6	22'	194.3
41'	104.4	32'	156.4	23'	195.0
42'	105.7	33'	157.3	24'	195.7
43'	106.9	34'	158.1	25'	196.4
44'	108.1	35'	158.9	26'	197.1
45'	109.4	36'	159.8	27'	197.7
46'	110.6	37'	160.6	28'	198.4
47'	111.8	38'	161.4	29'	199.1
48'	113.0	39'	162.3	30'	199.8
49'	114.1	40'	163.1	31'	200.4
50'	115.3	41'	163.9		
51'	116.4	42'	164.7		

TABLE No. 3.

Mr. Sundstrom.

x	y	l	R	α	x'	y'	l'
4.9	0.051	4.900	128.160	1° 05′ 43″	5.000	0.020	5.000
5.0	0.033	5.000	125.597	1° 08′ 25″	5.104	0.034	5.104
9.615	0.236	9.620	65.279	4° 12′ 35″	10.000	0.250	10.006
10.0	0.265	10.006	62.761	4° 33′ 08″	10.413	0.281	10.415
10.1	0.273	10.106	62.140	4° 38′ 36″	10.522	0.290	10.524
10.2	0.282	10.206	61.531	4° 44′ 07″	10.630	0.300	10.632
10.3	0.290	10.307	60.928	4° 49′ 42″	10.738	0.308	10.741
10.4	0.299	10.407	60.343	4° 55′ 19″	10.847	0.317	10.850
10.5	0.307	10.507	59.768	5° 01′ 00″	10.955	0.327	10.959
10.6	0.316	10.608	59.199	5° 06′ 44″	11.064	0.337	11.068
10.7	0.325	10.708	58.647	5° 12′ 31″	11.173	0.346	11.178
10.8	0.334	10.809	58.098	5° 18′ 20″	11.282	0.356	11.287
10.9	0.344	10.909	57.566	5° 24′ 14″	11.390	0.367	11.396
11.0	0.353	11.009	57.043	5° 30′ 10″	11.499	0.377	11.505
11.1	0.363	11.110	56.524	5° 36′ 10″	11.608	0.388	11.615
11.2	0.373	11.210	56.020	5° 42′ 13″	11.718	0.399	11.725
11.3	0.383	11.311	55.520	5° 48′ 18″	11.827	0.410	11.835
11.4	0.393	11.412	55.029	5° 54′ 27″	11.936	0.421	11.944
11.5	0.404	11.512	54.551	6° 00′ 39″	12.045	0.433	12.054
11.6	0.414	11.613	54.076	6° 06′ 54″	12.155	0.444	12.164
11.7	0.425	11.713	53.615	6° 13′ 13″	12.264	0.456	12.274
11.8	0.436	11.814	53.156	6° 19′ 34″	12.374	0.468	12.385
11.9	0.447	11.914	52.710	6° 25′ 58″	12.483	0.480	12.494
12.0	0.459	12.015	52.267	6° 32′ 26″	12.593	0.493	12.605
12.1	0.470	12.116	51.831	6° 38′ 57″	12.703	0.505	12.716
12.2	0.482	12.216	51.407	6° 45′ 30″	12.813	0.518	12.827
12.3	0.494	12.317	50.985	0° 52′ 07″	12.923	0.531	12.937
12.4	0.506	12.418	50.571	6° 58′ 47″	13.033	0.545	13.048
12.5	0.518	12.519	50.163	7° 05′ 29″	13.143	0.558	13.159
12.6	0.531	12.619	49.765	7° 12 15″	13.253	0.572	13.270
12.7	0.544	12.720	49.370	7° 19′ 04″	13.364	0.586	13.382
12.8	0.557	12.821	48.961	7° 25′ 56″	13.474	0.601	13.492
12.9	0.570	12.922	48.598	7° 32′ 51″	13.584	0.615	13.604
13.0	0.583	13.023	48.221	7° 39′ 49″	13.694	0.629	13.715
13.1	0.597	13.124	47.850	7° 46′ 50″	13.805	0.645	13.827
13.2	0.610	13.225	47.485	7° 53′ 54″	13.915	0.659	13.938
13.3	0.624	13.326	47.125	8° 01′ 00″	14.026	0.675	14.050
13.4	0.639	13.427	46.770	8° 08′ 10″	14.137	0.691	14.162
13.5	0.653	13.528	46.421	8° 15′ 23″	14.248	0.707	14.274
13.6	0.668	13.629	46.077	8° 22′ 39″	14.359	0.723	14.386
13.7	0.682	13.730	45.738	8° 29′ 57″	14.470	0.739	14.498
13.8	0.697	13.831	45.404	8° 37′ 19″	14.581	0.755	14.611
13.9	0.713	13.932	45.075	8° 44′ 44″	14.692	0.773	14.723
14.0	0.728	14.033	44.751	8° 52′ 11″	14.803	0.790	14.835
14.1	0.744	14.134	44.431	8° 59′ 41″	14.914	0.808	14.948
14.2	0.760	14.236	44.113	9° 07′ 15″	15.025	0.826	15.060
14.3	0.776	14.337	43.802	9° 14′ 51″	15.136	0.844	15.173

Mr. Sundstrom.

TABLE No. 3—(*Concluded*).

x	y	l	R	a	x'	y'	l'
14.4	0.792	14.439	43.492	9° 22' 30"	15.248	0.862	15.286
14.5	0.809	14.540	43.190	9° 30' 12"	15.359	0.880	15.398
14.6	0.826	14.641	42.892	9° 37' 56"	15.471	0.899	15.512
14.7	0.843	14.743	42.596	9° 45' 44"	15.583	0.918	15.626
14.8	0.860	14.844	42.306	9° 53' 34"	15.695	0.937	15.739
14.9	0.878	14.946	42.017	10° 01' 27"	15.807	0.957	15.853
15.0	0.896	15.047	41.735	10° 09' 23"	15.918	0.977	15.966
15.1	0.914	15.149	41.454	10° 17' 22"	16.030	0.997	16.080
15.2	0.932	15.250	41.179	10° 25' 24"	16.142	1.017	16.193
15.3	0.951	15.352	40.906	10° 33' 28"	16.254	1.038	16.307
15.4	0.969	15.454	40.636	10° 41' 35"	16.366	1.059	16.421
15.5	0.988	15.556	40.369	10° 49' 45"	16.478	1.080	16.535
15.6	1.008	15.657	40.109	10° 57' 57"	16.590	1.103	16.650
15.7	1.027	15.759	39.849	11° 06' 12"	16.702	1.124	16.764
15.8	1.047	15.861	39.593	11° 14' 30"	16.815	1.147	16.879
15.9	1.067	15.963	39.340	11° 22' 51"	16.927	1.169	16.993
16.0	1.087	16.065	39.090	11° 31' 15"	17.039	1.191	17.107
16.1	1.108	16.167	38.844	11° 39' 40"	17.152	1.214	17.222
16.2	1.128	16.269	38.600	11° 48' 12"	17.265	1.238	17.338
16.3	1.149	16.372	38.357	11° 56' 40"	17.378	1.262	17.454
16.4	1.171	16.474	38.120	12° 05' 13"	17.490	1.286	17.568
16.5	1.192	16.576	37.885	12° 13' 50"	17.603	1.310	17.684
16.6	1.214	16.678	37.653	12° 22' 31"	17.716	1.335	17.799
16.7	1.236	16.781	37.422	12° 31' 10"	17.829	1.360	17.915
16.8	1.258	16.883	37.196	12° 39' 54"	17.942	1.385	18.031
16.9	1.281	16.986	36.971	12° 48' 41"	18.055	1.411	18.147
17.0	1.304	17.088	36.750	12° 57' 30"	18.168	1.437	18.263
17.1	1.327	17.191	36.530	13° 06' 21"	18.281	1.463	18.379
17.2	1.350	17.294	36.312	13° 15' 15"	18.394	1.489	18.495
17.3	1.374	17.396	36.099	13° 24' 12"	18.507	1.516	18.611
17.4	1.398	14.499	35.887	13° 33' 11"	18.620	1.543	18.727
17.5	1.422	17.603	35.675	13° 42' 12"	18.733	1.570	18.843
17.6	1.447	17.706	35.467	13° 51' 16"	18.846	1.598	18.959
17.7	1.472	17.809	35.262	14° 00' 22"	18.960	1.626	19.077
17.8	1.497	17.912	35.060	14° 09' 30"	19.074	1.655	19.194
17.9	1.522	18.015	34.859	14° 18' 41"	19.187	1.683	19.311
18.0	1.548	18.118	34.661	14° 27' 54"	19.300	1.712	19.427
18.1	1.574	18.222	34.463	14° 37' 10"	19.414	1.741	19.545
18.2	1.600	18.325	34.269	14° 46' 27"	19.528	1.771	16.663
18.3	1.627	18.428	34.078	14° 55' 48"	19.642	1.802	19.781
18.4	1.653	18.532	33.887	15° 05' 10"	19.756	1.832	19.899
18.5	1.680	18.636	33.697	15° 14' 34"	19.869	1.863	20.016
18.6	1.708	18.739	33.512	15° 24' 01"	19.983	1.895	20.134
18.7	1.736	18.843	33.327	15° 33' 10"	20.097	1.927	20.253
18.8	1.764	18.947	33.144	15° 43' 01"	20.211	1.959	20.371
18.9	1.792	19.051	32.964	15° 52' 35"	20.325	1.991	20.490
19.0	1.820	19.155	32.784	16° 02' 10"	20.439	2.023	20.608

Example.—Suppose it is required to place a curve of 39 ft. radius, with transition approaches, between two tangents, making an angle of 41° 10'. The nearest radius to 39, in Table No. 3, is 39.09, which in this case must be used. The table gives everything but the tangent. This is found by the following formula: $T = (R + s)$ tan. $\frac{1}{2} I + x - R$ sin. a; s being equal to $\frac{y}{4}$. Then we have $T = \left(39.09 + \frac{1.087}{4}\right)$ tan. 20° 35' + 16.0 — 39.09 sin. 11° 31' 15" = 14.782 + 16.000 — 7.807 = 22.975. From inspection of Table No. 3 it will be found that the radius of curvature gradually decreases; thus it is, at the distance of 5 ft. from the point of curve, = 125.597; at the distance of 10 ft. = 62.761, etc. When $I = 2a$, the curve will be composed of the two approaches only. If the radius selected produces an angle, a, so large that $2a > I$, a larger radius must be selected, so as to get a correspondingly smaller angle a, or so that $2a$ becomes smaller than or equal to I.

Mr. Wentworth.

CHARLES C. WENTWORTH, M. Am. Soc. C. E. (by letter).—When a car leaves a tangent and enters a curve there are two distinct motions imparted to it: First, one of translation about the center of the curve; second, one of rotation about its own center.

For the first of these motions, and they may be considered separately, a circular curve is as good as any other. In fact, any curve is, at a given point, circular, with its corresponding radius of curvature. All that is necessary to be done, then, in treating this motion, is to elevate the outer rail so that the resultant of the centrifugal force and gravity may be as near as desired to a perpendicular to the plane of the track.

To find the theoretical elevation in inches for a track of standard gauge, the formula $E = \frac{v^2 D}{1\,500}$ may be used, in which v is the speed in miles per hour and D the degree of curvature. This is merely a reduction from the usual formulas which are given in inconvenient units, and is put in form for use on the slide rule.

The second of the two motions before mentioned is that which causes the jar on entering or leaving an accurately laid out circular curve. This motion is precisely the same as if the car were mounted on a frictionless turn-table and a force applied to turn it. Applying such a force, sufficient to start the table from a state of rest to, say, an angular motion of 1° in 1 second, the same force would turn the car through a total angle of 4° in 2 seconds. This is what should be accomplished by a transition curve. The condition, then, of such a curve is that its tangential angle at any point shall be proportional to the square of the length of the curve, from the point of beginning, or *P. C.*, to the point in question.

The reason why the putting in of transition curves is left to the track foreman, who invariably does it, rather than to the engineer of location, appears to be that such curves have been unnecessarily loaded with too much mathematics for a busy man. They can be run in on the ground from either end without the use of any tables whatever.

For instance, it may be desired to use a transition curve such that the deflection for the first station of 100 ft. will be 25′ from the tangent at the *P. C.* The deflection for 2 stations, or 200 ft., will be four times this, or 1° 40′; for $2\frac{1}{2}$ stations, or 250 ft. $(2\frac{1}{2})^2 \times 25'$, or 2° 36′. For any distance, the square of that distance, in feet, multiplied by 0.0025, will be the deflection, in minutes, from the original tangent.

After running the transition curve as far as desired, move the transit to the end of it, take a backsight on the *P. C.* and turn an additional angle equal to twice the deflection angle used in setting the stake at the end of the transition curve. This will put the instrument on tangent at the end of the transition curve; the tangential angle at that

Mr. Wentworth. or any point being three times the deflection used in setting such point.

The deflection for the first 100 ft. being assumed as 25′, the degree of curvature at Station 1 (100 ft. from the *P. C.*) will be 6 times 25′, or 2° 30′. This is the basis of the degree of curvature at any point. At 200 ft. from the *P. C.*, it is 5°, and at 300 ft., 7° 30′. At any point it is 6 times the deflection for the first 100 ft. multiplied by the length of the curve, in stations, to the point in question.

After running in the main curve, the *P. T.* of the transition curve can be set by using, as a deflection from tangent, twice the deflection which would be needed to set the point at the end of the main curve from the *P. T.* Then, after moving up to the *P. T.*, the intervening stakes can be set back of the transit.

Of course, any deflection may be assumed for the first 100 ft., instead of 25′, in order to make the maximum transition curve of any desired length. Mr. Sundstrom's tables agree closely with the figures obtained as described for a transition curve, the deflection for which for the first 100 ft. is 37.6′.

The method of planning and locating transition curves given herein is inexact only in the tangential angles when the curve is of considerable length. For instance, if 12° be the degree of the main curve and the transition curve be 200 ft. long (1° deflection for the first 100 ft.), there will be no error. If 300 ft. long (40′ deflection for the first 100 ft.) the error is 2′. If 400 ft. long (30′ deflection for the first 100 ft.) the error is 5′. Nevertheless, as the total angle is correctly carried through from tangent to tangent, and as these small differences in no way affect the track, the error, if it may be so called, is immaterial. The greatest error in each case is given above, which occurs at the end of the transition curve.

Mr. Boggs. J. I. Boggs, Assoc. M. Am. Soc. C. E. (by letter).—Transition curves are undoubtedly of some advantage where the radius is less than 600 ft., but the writer heartily agrees with the "practical" man, that they are a "needless refinement" where the radius of curvature exceeds that figure. The elevation of the outer rail is the determining factor in the easement, and it is a well-known fact that if this elevation is properly adjusted for speed and gauge, and graded back on the tangent the length of the easement, no shock will be felt upon entering the curve. The great trouble is the proper adjustment of the elevation for the varying conditions of speed and grade found on railroads. To attempt, however, to educate section men up to the niceties of the cubic parabola appears to the writer a hopeless and equally useless task. The practice of running the elevation back on the tangent from full elevation at the point of curve to zero at the point of easement, or as near thereto as practicable, is much more simple, answers every purpose of the spiral, and is readily understood by the average foreman.

As stated, the elevation of the outer rail is the governing factor, and this same elevation presents, at times, anomalies, refusing to be bound by printed rules or mathematical equations. The writer had two 2° curves to contend with, that had always given considerable trouble. One was on a 66-ft. grade, the other was on a level at the foot of two 53-ft. grades. The rule of the road was 1 in. elevation for each degree of curvature, or 2 ins. in these cases, while the formula, $l = \frac{G\,r^2}{32.2\,R}$, gave $2\frac{5}{16}$ ins.; neither would do. Finally, the curve on the 66-ft. grade was elevated to $2\frac{1}{2}$ ins. and the elevation graded out on the tangent for 100 ft., as near as practicable; the other was elevated $1\frac{3}{4}$ ins. and graded out on the tangent for 70 ft. No further trouble was experienced with either of them, and the writer has long since arrived at the conclusion that a study of the idiosyncrasies of each particular curve is of far more benefit than all the tables printed. Mr. Boggs.

The writer's practice in staking out transition curves has been to use a simple offset, measured in with the tape line, and determined from the equation, $y = \frac{x^3}{6\,P}$. Henck's Field Book* contains all the information any engineer requires in staking out a transition curve, and Mr. Sundstrom's demonstration is but a reprint of these four pages. The writer thinks Mr. Snyder's cost, $28.27 per mile, would be excessive if applied to railways.

Extreme mathematical nicety in actual practice only serves to multiply useless labor, and increases the pay-roll without adding to the efficiency of the service. For instance, the writer has seen the end-areas of cross-sections calculated to the nearest one-hundredth of a square foot and the contents of the prism figured to a minimum by the old prismoidal formula; while, in the field work, it was permissible to hold the rod within two-tenths, and drive the stake within 6 ins. of the theoretical point. The writer digresses here simply to illustrate a "needless refinement." He holds that the nearest square foot and an average of the end-areas would have been sufficient, and, probably, would have resulted in a saving of $1 200 per annum on the pay-roll. The same thing applies, as a rule, to all this curvature discussion. Engineers are busy figuring, in the office, to the nearest one-hundredth of an inch on grades and alignments, wasting time and paper, and absorbing the stockholders' dividends in needless discussions; while on the track itself the inch which has been so laboriously figured out cannot be found, let alone the one-hundredth part of it. Curvature is pretty much of a bugaboo, anyway. With good track and good rolling stock, the traveling around curves is easy; bad track and poor equipment gives rough traveling, even on tangents.

* Edition of 1881, pp. 125 to 128, inclusive.

Mr. Alden. CHARLES A. ALDEN, Assoc. M. Am. Soc. C. E. (by letter). – Transition curves for street-railway track are now almost universally used in the United States. The cubic parabola, or its variations, would answer as well for street railways as for steam railroads, as far as theoretical considerations of alignment are concerned. But the radii of curves for the former are generally small and the transition short, requiring the rails to be curved to a template, and by machine. It is also necessary to make provision for the use of tongue-switches, as it is impracticable to make a wholly satisfactory switch having a radius greater than about 100 ft., although a certain type of switch can be made of about 200 ft. radius, having, however, certain mechanical disadvantages.

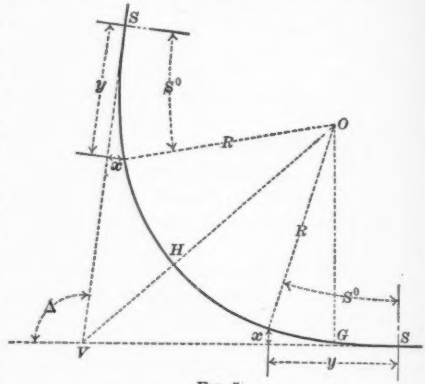


FIG. 5.

Given: A circular curve with symmetrical spirals; to find the tangent and external distances.

$O\,G = R + x -$ ver. sin. $S°\,R$;
$G\,S = y -$ sin. $S°\,R$;
Tan. Dist. $= O\,G$ tan. $\frac{1}{2}\Delta + G\,S$;
Ex. Dist. $= O\,G$ ex-sec. $\frac{1}{2}\Delta + x -$ ver. sin. $S°\,R$.

On the somewhat rare occasions when the spirals are involved in the frog work, the calculations of intersections of cubic parabolas with each other or with a circular curve, would be quite complex, as compared with circular curves.

The calculations, in the latter case, simply resolve themselves into the solution of oblique triangles, readily adapted to logarithmic solution.

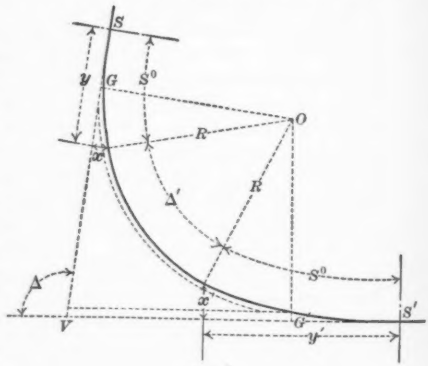


Fig. 6.

General solution for unsymmetrical curves.

$O\,G = R + x - R$ ver. sin. $S°$;
$G\,S = y - R$ sin. $S°$;
$O\,G' = R + x' - R$ ver. sin. $S°'$;
$G'\,S' = y' - R$ sin. $S°'$;
$V\,S =$ tan. $\frac{1}{2}\Delta\ O\,G + G\,S + \frac{O\,G' - O\,G}{\text{sin. }\Delta}$.
$V\,S' =$ tan. $\frac{1}{2}\Delta\ O\,G + G'\,S' \pm \frac{O\,G' - O\,G}{\text{tan. }\Delta}$.

It is necessary to have a series of transitions in order to suit the different radii met with in practice, and, at the same time, it is desirable to limit the number to those

absolutely necessary in order to avoid the expense and delay incident to calculating, laying out and preparing templates for a new transition curve, for each curve built. Mr. Alden.

Figs. 5 and 6, and Table No. 4, which explain themselves, are those used by the company with which the writer is connected. During the past year about 400 curves were built to these standards. The easements for 200-ft. radius switches are intended for, and generally used for, a Y or equilateral switch.

The other two principal manufacturers of street-railway material have standard transitions of similar form.

The writer estimates that from 70 to 80% of street-railway curves are now built according to this general form of transition.

TABLE No. 4.

Rad.	Angle.	x.	y.	$S°$	Ver. Sine.	Sine.
			Spiral No. 2.			
300	0° 30′	0.011	2.618	0° 30′	0.00004	0.00873
150	1° 00′	0.057	5.235	1° 30′	0.00034	0.02618
100	1° 30′	0.160	7.851	3° 00′	0.00137	0.05234
75	2° 00′	0.342	10.463	5° 00′	0.00381	0.08716
60	2° 30′	0.627	13.065	7° 30′	0.00856	0.13053
50	3° 00′	1.036	15.651	10° 30′	0.01675	0.18224
42½	3° 30′	1.587	18.187	14° 00′	0.02970	0.24192
37½	4° 00′	2.309	20.703	18° 00′	0.04894	0.30902
			Switch Easement, *S.* 2–75.			
75	7° 50′	0.700	10.222	7° 50′	This easement gives an *O. G.* equal to, and a *G. S.* 3.346 less than, Spiral No. 2.	
45⅓	2° 50′	1.036	12.305	10° 30′		
42½	3° 30′	1.587	14.841	14° 00′		
37½	4° 00′	2.309	17.357	18° 00		
			Spiral No. 2½.			
444	0° 20′	0.007	2.583	0° 20′	0.00002	0.00582
222	0° 40′	0.038	5.166	1° 00′	0.00015	0.01745
148	1° 00′	0.105	7.748	2° 00′	0.00061	0.03490
111	1° 20′	0.226	10.328	3° 20′	0.00169	0.05814
89	1° 40′	0.414	12.910	5° 00′	0.00381	0.08716
74	2° 00′	0.684	15.478	7° 00′	0.00745	0.12187
63½	2° 20′	1.051	18.038	9° 20′	0.01324	0.16218
55½	2° 40′	1.529	20.576	12° 00′	0.02185	0.20791
49	3° 00′	2.128	23.070	15° 00′	0.03407	0.25882
44½	3° 20′	2.870	25.550	18° 20′	0.05076	0.31454
40½	3° 40′	3.763	27.983	22° 00′	0.07282	0.37461
			Switch Easement, *S.* 2½–100.			
102⅓	6° 30′	0.658	11.584	6° 30′	This easement gives an *O. G.* equal to, and a *G. S.* 3.640 less than, Spiral No. 2½.	
56½	5° 30′	1.529	16.936	12° 00′		
49	3° 00′	2.128	19.430	15° 00′		
44½	3° 20′	2.870	21.910	18° 20′		
40½	3° 40′	3.763	24.343	22° 00′		
			Switch Easement, *S.* 2½–200.			
200	4° 0′	0.487	13.951	4° 0′	This easement gives an *O. G.* 0.250 greater, and a *G S.* 3.286 greater, than Spiral No. 2½.	
129	1° 0′	0.664	16.196	5° 0′		
74	2° 0′	0.934	18.764	7° 0′		
63½	2° 20′	1.301	21.324	9° 20′		
55½	2° 40′	1.779	23.862	12° 00′		
49	3° 00′	2.378	26.356	15° 00′		
44½	3° 20′	3.120	28.836	18° 20′		
40½	3° 40′	4.013	31.269	22° 00′		

NOTE.—All dimensions are for the center line of the track.

Mr. Alden.

TABLE No. 4—(*Continued*).

Rad.	Angle.	x.	y.	$S°$	Ver. Sine.	Sine.
			Spiral No. 3.			
300	1° 00′	0.046	5.236	1° 00′	0.00015	0.01745
150	2° 00′	0.229	10.468	3° 00	0.00137	0.05234
100	3° 00′	0.639	15.688	6° 00′	0.00548	0.10453
75	4° 00′	1.368	20.871	10° 00′	0.01519	0.17365
60	1° 00′	2.501	25.982	15° 00′	0.03407	0.25882
50	6° 00′	4.118	30.959	21° 00′	0.06642	0.35837
40	7° 00′	6.143	35.403	28° 00′	0.11705	0.46947
			Switch Easement, *S.* 3-100.			
102⅓	6° 30′	0.658	11.584	6° 30′	This easement gives an *O. G.* equal to, and a *G. S.* 4.391 less than, Spiral No. 3.	
81	3° 30′	1.368	16.480	10° 00′		
60	5° 00′	2.501	21.591	15° 00′		
50	6° 00′	4.118	26.568	21° 00′		
40	7° 00′	6.143	31.012	28° 00′		
			Switch Easement, *S.* 3-200.			
200	4° 00′	0.487	13.951	4° 00′	This easement gives an *O. G.* 0.250 greater, and a *G. S.* 2.853 greater, than Spiral No. 3.	
132	2° 00′	0.889	18.541	6° 00′		
75	4° 00′	1.618	23.724	10° 00′		
60	5° 00′	2.751	28.835	15° 00′		
50	6° 00′	4.368	33.812	21° 00′		
40	7° 00′	6.393	38.256	28° 00′		
			Spiral No. 4.			
420	0° 42′	0.031	5.131	0° 42′	0.00007	0.01222
210	1° 24′	0.157	10.261	2° 06′	0.00067	0.03664
140	2° 06′	0.439	15.384	4° 12′	0.00269	0.07324
105	2° 48′	0.939	20.490	7° 00′	0.00745	0.12187
84	3° 30′	1.720	25.561	10° 30′	0.01675	0.18224
70	4° 12′	2.839	30.567	14° 42′	0.03273	0.25376
60	4° 54′	4.352	35.469	19° 36′	0.05794	0.33545
			Switch Easement, *S.* 4-200.			
200	4° 00′	0.487	13.951	4° 00′	This easement gives an *O. G.* 0.178 greater, and a *G. S.* equal to, Spiral No. 4.	
125½	3° 00′	1.117	20.490	7° 00′		
84	3° 30′	1.898	25.561	10° 30′		
70	4° 12′	3.017	30.567	14° 42′		
60	4° 54′	4.530	35.469	19° 36′		
			Spiral No. 5.			
600	0° 30′	0.023	5.236	0° 30′	0.00004	0.00873
300	1° 00′	0.114	10.471	1° 30′	0.00034	0.02618
200	1° 30′	0.320	15.703	3° 00′	0.00137	0.05234
150	2° 00′	0.685	20.926	5° 00′	0.00381	0.08716
120	2° 30′	1.255	26.130	7° 30′	0.00856	0.13053
100	3° 00′	2.073	31.302	10° 30′	0.01675	0.18224
85	3° 30′	3.175	36.374	14° 00′	0.02970	0.24192
			Switch Easement, *S.* 5-200.			
200	4° 00′	0.487	13.951	4° 00′	This easement gives an *O. G.* equal to, and a *G. S.* 4.468 less than, Spiral No. 5.	
144	1° 00′	0.685	16.458	5° 00′		
120	2° 30′	1.255	21.662	7° 30′		
100	3° 00′	2.073	26.834	10° 30′		
85	3° 30′	3.175	31.906	14° 00′		
			Spiral No. 6.			
900	0° 20′	0.015	5.236	0° 20′	0.00002	0.00582
450	0° 40′	0.076	10.472	1° 00′	0.00015	0.01745
300	1° 00′	0.213	15.706	2° 00′	0.00061	0.03490
225	1° 20′	0.457	20.936	3° 20′	0.00169	0.05814
180	1° 40′	0.837	26.158	5° 00′	0.00381	0.08716
150	2° 00′	1.385	31.365	7° 00′	0.00745	0.12187
128	2° 20′	2.125	36.524	9° 20′	0.01324	0.16218

NOTE.—All dimensions are for the center line of the track.

TABLE No. 4—(*Concluded*).

Mr. Alden.

Rad.	Angle.	*x*.	*y*.	*S*°	Ver. Sine.	Sine.
			Switch Easement, *S*. 6-200.			
200	4° 00′	0.487	13.951	4° 00′	This easement gives an *O. G.* equal to, and a *G. S.* 7.770 less than, Spiral No. 6.	
255	1° 00′	0.837	18.388	5° 00′		
150	2° 00′	1.385	23.595	7° 00′		
128	2° 20′	2.125	28.754	9° 20′		
			Spiral No. 7.			
1 260	0° 15′	0.012	5.498	0° 15′	0.00001	0.00436
630	0° 30′	0.060	10.995	0° 45′	0.00009	0.01309
420	0° 45′	0.168	16.492	1° 30′	0.00034	0.02618
315	1° 00′	0.360	21.987	2° 30′	0.00095	0.04362
252	1° 15′	0.660	27.475	3° 45′	0.00214	0.06540
210	1° 30′	1.091	32.957	5° 15′	0.00420	0.09150
180	1° 45′	1.678	38.424	7° 00′	0.00745	0.12187
157	......	.....	......	......	....	....
			Spiral No. 8.			
1 890	0° 10′	0.008	5.498	0° 10′	0.00000	0.00291
945	0° 20′	0.040	10.996	0° 30′	0.00004	0.00873
630	0° 30′	0.112	16.493	1° 00′	0.00015	0.01745
472½	0° 40′	0.241	21.990	1° 40′	0.00042	0.02908
378	0° 50′	0.441	27.483	2° 30′	0.00095	0.04362
315	1° 00′	0.729	32.973	3° 30′	0.00187	0.06105
270	1° 10′	1.120	38.457	4° 40′	0.00332	0.08136
236	......	.....	......	......	....	....
			Spiral No. 9.			
2 730	0° 7′	0.006	5.559	0° 7′	0.00000	0.00204
1 365	0° 14′	0.028	11.118	0° 21′	0.00002	0.00611
910	0° 21′	0.079	16.677	0° 42′	0.00007	0.01222
682½	0° 28′	0.170	22.234	1° 10′	0.00021	0.02036
546	0° 35′	0.311	27.791	1° 45′	0.00047	0.03054
455	0° 42′	0.515	33.346	2° 27′	0.00091	0.04275
390	0° 49′	0.792	38.899	3° 16′	0.00162	0.05698
341	......	.....	......	......	....	....
			Spiral No. 10.			
3 780	0° 5′	0.004	5.498	0° 5′	0.00000	0.00145
1 890	0° 10′	0.020	10.996	0° 15′	0.00001	0.00436
1 260	0° 15′	0.056	16.493	0° 30′	0.00004	0.00873
945	0° 20′	0.120	21.991	0° 50′	0.00011	0.01454
756	0° 25′	0.220	27.488	1° 15′	0.00024	0.02181
630	0° 30′	0.364	32.983	1° 45′	0.00047	0.03054
540	0° 35′	0.560	38.478	2° 20′	0.00083	0.04071
472	......	.....	......	......	......	....
			Spiral No. 11.			
5 250	0° 4′	0.0035	6.109	0° 4′	0.00000	0.00116
2 625	0° 8′	0.0178	12.217	0° 12′	0.00001	0.00349
1 750	0° 12′	0.0498	18.326	0° 24′	0.00002	0.00698
1 312½	0° 16′	0.1066	24.434	0° 40′	0.00007	0.01164
1 050	0° 20′	0.1955	30.542	1° 0′	0.00015	0.01745
875	0° 24′	0.3234	36.649	1° 24′	0.00030	0.02443
750	0° 28′	0.4975	42.756	1° 52′	0.00053	0.03257
656	......	......	......	......	......	....
			Spiral No. 12.			
7 140	0° 3′	0.0027	6.231	0° 3′	0.00000	0.00087
3 570	0° 6′	0.0136	12.462	0° 9′	0.00000	0.00262
2 380	0° 9′	0.0381	18.692	0° 18′	0.00001	0.00524
1 785	0° 12′	0.0816	24.923	0° 30′	0.00004	0.00873
1 428	0° 15′	0.1495	31.153	0° 45′	0.00009	0.01309
1 190	0° 18′	0.2474	37.384	1° 3′	0.00017	0.01832
1 020	0° 21′	0.3806	43.613	1° 24′	0.00030	0.02443
892	......	......	......	......	....	....

NOTE.—All dimensions are for the center line of the track.

Mr. Howe. HORACE J. HOWE, M. Am. Soc. C. E. (by letter).—It is somewhat appalling to the busy engineer to take up each new article on the transition curve, unless he has time to go into the mathematics and see that the principles are few compared to the applications. The author has given an illustration of this by taking the well-known cubic parabola, and deriving formulas which he has found useful in his practice.

The writer's experience has been mostly on steam railroads, and, looking back at the conditions under which he labored, and under which he believes the majority now labors, it seems proper to restate the well-known but scattered conclusions which govern the practical staking out of a transition curve; to the end that all theorizers may profit thereby.

The cost of staking out should be reduced to a minimum, and, consequently, the simplest possible curve should be used.

The writer's first attempt was a series of chords or arcs beginning with a 1° curve and working up to the degree desired. Many bright engineers to-day postpone the evil day, by reserving this right to themselves. The time consumed, however, is prohibitory.

About this time, Searles published his tables, and used successive chords, with radii constantly diminishing for each chord. The objection is the disjointed character of the curve, and the labor of calculation. The treatment is too general for the special work referred to.

Next came the hyperbolic or Holbook spiral, which did specialize, with tables for only two or three cases. It had several advantages besides. The deflections vary about as the square of the distance. The backsight from the point of circular curve is double the foresight.

x_0 is one-fourth of x (nearly).

y_0 is one-half of y (nearly).

Bearing these properties in mind, it is quite possible to do without tables in the field. Can the cubic parabola do better than this?

So far as riding qualities are concerned, one easement curve is as good as another. References are made to the "true" transition curve and to the "theoretical" transition curve; but the running of a pair of wheels around a superelevated track is too complicated an operation to allow of any such claim. A good job could be done by one of half a dozen methods, and the experts would never know the difference.

Transition curves, then, should be of the simplest. They should be designed on the principle that maintenance engineers have no expense accounts; and, on that other principle, that the mental performance of any man when subject to conditions of weather is inferior to that of the same man indoors; finally, that trackmen do not throw track very often (not so often, generally, as the engineer stakes it out), and that the engineer must refresh his memory, perhaps at yearly intervals, on the subject of transition curves.

W. B. LEE, M. Am. Soc. C. E. (by letter).—To the engineer who has occasion to solve track problems in transitions, the following equations for the inside and outside rails of a curved track, the center line of which is a cubic parabola, may prove both useful and interesting: Mr. Lee.

Let $y = \frac{x^3}{6\,C}$ be the equation of the center line, g the gauge of track, y_i and y_o the ordinates of points on the inside and outside rails from the initial tangents through their own points of transition curve (*P. T. C.*), respectively; x remaining the same.

Then
$$y_i = \frac{x^3}{6\,C} + \frac{g\,x^4}{16\,C^2},$$
and
$$y_o = \frac{x^3}{6\,C} - \frac{g\,x^4}{16\,C^2}.$$

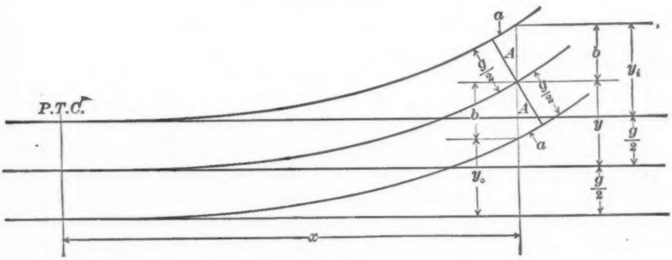


FIG. 7.

Referring to Fig. 7, they may be demonstrated as follows:

$$y_i = y + b - \frac{g}{2} \dots\dots\dots\dots (1)$$

By the same course of reasoning as followed on page 380:

$$b - \frac{g}{2} = \frac{a^2}{2 \times \frac{g}{2}} = \frac{a^2}{g} \dots\dots\dots\dots (2)$$

In the triangle shown in Fig. 7,

$$a = \frac{g}{2} \tan A \dots\dots\dots\dots (3)$$

from the equation of the cubic parabola,

$$\tan. A = \frac{x^2}{2\,C} \dots\dots\dots\dots (4)$$

substituting (4) in (3),

$$a = \frac{g\,x^2}{4\,C} \dots\dots\dots\dots (5)$$

and

$$a^2 = \frac{g^2\,x^4}{16\,C^2} \dots\dots\dots\dots (6)$$

Mr. Lee. substituting (6) in (2) we have

$$b - \frac{g}{2} = \frac{g\,x^4}{16\,C^2} \quad \ldots\ldots\ldots\ldots (7)$$

substituting (7) in (1),

$$y_i = y + \frac{g\,x^4}{16\,C^2} = \frac{x^3}{6\,C} + \frac{g\,x^4}{16\,C^2} \quad \ldots\ldots\ldots\ldots (8)$$

From Fig. 7, and similar reasoning,

$$y_o = y + \frac{g}{2} - b = y - \left(b - \frac{g}{2}\right) = \frac{x^3}{6\,C} - \frac{g\,x^4}{16\,C^2}.$$

The deflection angles, B_i and B_o, are obtained by dividing through by x, which gives

$$\tan.\ B_i = \frac{x^2}{6\,C} + \frac{g\,x^3}{16\,C^2};$$

$$\tan.\ B_o = \frac{x^2}{6\,C} - \frac{g\,x^3}{16\,C^2}.$$

The total angle, A, is found by differentiating;

whence,

$$\tan.\ A_i = \frac{x^2}{2\,C} + \frac{g\,x^3}{4\,C^2}.$$

$$\tan.\ A_o = \frac{x^2}{2\,C} - \frac{g\,x^3}{4\,C^2}.$$

It will be observed that these formulas are approximations, but, as the locations obtained from them will not vary from theoretical values more than $\frac{1}{32}$ in. in extreme cases, their practical utility is unimpaired; moreover, the gauge of the track is still exact.

In conclusion, the writer would make the following suggestions as to the selection and use of transition curves. Where there are no turnouts or crossings, and it is desired to stake out the curve with a transit, he would recommend the selection of the formula $y = \frac{s^3}{6\,C}$; for, by this, the distances around the curve may be multiples of some convenient number as 5, 10 or 25 ft.; and, as the transitman is usually more intelligent and experienced than the chainman, the former may be more readily trusted to deflect fractional angles than the latter to measure fractional distances.

For turnouts and crossings $y = \frac{x^3}{6\,C}$ is more convenient, and in switches the equation of the center line will do for both inside and outside rails. Indeed, in a simple turnout, it will also serve to give the frog lead within 3 to 6 ins. of the correct position, which is close enough for practical purposes. However, where a complicated track system exists, or it is desired to compute a set of crossing frogs, the equations for the inside and outside rails already given may be used to supplement the equation for the center line.

The writer cannot resist the temptation to digress from the subject, here, to corroborate the views expressed by Mr. Snyder, as to

Mr. Lee.

the practical utility of the slide rule in engineering computations. The writer has used one constantly for the past six years and has found new uses for it daily, and new beauties in its use. At first it did not seem to be adapted to the class of work in which he was engaged, but a study of both its limits and possibilities has enabled him to use it in the majority of his computations. It relieves entirely the mental drudgery of many complicated problems, as the expert performs the several operations almost mechanically. The writer has passed through the "Mannheim" and "Duplex" to the "Universal," and he would not set a price upon his instrnment if it could not be duplicated.

This last opportunity is taken to enter a solemn protest against the use of multiple compound curves as transitions. During the summer of 1900 the writer was required to design the track work for the Boston Elevated Railroad, on which no less than thirteen different tables of multiple compound curves were in use. The memory of those computations is a hideous nightmare. One crossing alone required 1 170 sq. ins. of figures before the gauge lines could be laid down on paper.

Referring to Mr. Wentworth's discussion, too much mathematics may be an excuse for a busy "man" but not for a busy engineer. It is true that theoretical mathematicians have loaded many simple problems with a mass of unnecessary refinements; but if the engineer is well grounded in his mathematics he will be able to unearth the precious scientific truth. Popularly considered an exact science, mathematics has little more claim to such distinction than several other physical sciences, and, as all the works of man are incomplete, so mathematics shares the faults and imperfections of human nature. The writer also agrees with Mr. Boggs, that "extreme mathematical nicety in actual practice only serves to multiply useless labor," but what is permissible variation in field work would be gross error in the shop. Each engineer must study the conditions of his own work, and make his computations accordingly. It appears that the engineers engaged in this discussion have done so, as each one's idea of accuracy seems to conform to his specialty.

The writer will leave to those who have the opportunity to observe actual conditions, the question of the necessity of transition curves, but, if railroad engineers insist upon their use, let them furnish the shop men with the simplest and most practical alignment.

The complication of trackwork, needlessly, is a too-frequent practice, among railroad engineers, when a little study might simplify it. For instance, on a scale of 40 ft. to the inch it is easy to take a curved ruler and draw a crossing, introducing a set of slip switches, but it is quite another matter to compute, detail, and manufacture this work. The cost of drawings and labor in the shop is doubled, and the maintenance of the track is increased considerably by reason of the

Mr. Lee. curves. A slight change of alignment on either side of such a crossing might allow it to be straight, and thus obviate these difficulties.

The writer is glad to know, from Mr. Alden's discussion, that the use of transition curves in frog work is rare, but cannot agree with him, that "the calculation of intersections of cubic parabolas with each other or with a circular curve, would be quite complex as compared with circular curves." The writer knows whereof he speaks when he makes the statement that the computation of frog work in transitions made up of circular curves is very complex, and believes that in the paper he has partially demonstated the fact that cubic parabolas are much simpler. Should a cubic parabola intersect a circle, the equation of the latter might be taken as $y = \frac{x^2}{2\ R}$, except for large values of x, when it might be necessary to use the exact equation $y = R - \sqrt{R^2 - x^2}$, or $2\ Ry - y^2 = x^2$.

Mr. Sundstrom's formula for rectifying the cubic parabola is the same as the writer's, but has been deduced in a more orthodox manner.

A table of offsets from a tangent to a cubic parabola may be obtained with a minimum amount of labor by observing the law of the series. Such a list forms a cubic series, of which the following fundamental series, the first term of which is 1, is an example:

Series............................	1		8		27		64		125
First order of differences...........		7		19		37		61	
Second order of differences.........			12		18		24		
Third order of differences..........				6		6			
Fourth order of differences					0				

It is only necessary to compute the first term of the series. Then the first term of the first order of differences is seven times the first term of the series; the first term of the second order of differences is twelve times the first term of the series; and all the third differences are six times the first term of the series. This completes the first inclined column, from which, by addition, as many offsets may be obtained as are required. The writer* uses similar methods for marking templets of circular curves, calling the equation of a circle $y = \frac{x^2}{2\ R}$.

* *Engineering News*, April 29th, 1897.